MODAL ANALYSIS OF STRUCTURES BY AN ITERATIVE RAYLEIGH-RITZ TECHNIQUE

By

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SUMMARY

A method of modal analysis of structures is presented. The method is based on repeated applications of the well known Rayleigh-Ritz technique using improved coordinate vectors for each iteration. This procedure is convergent, since each application of the Rayleigh-Ritz technique yields a better approximation of the modal properties of the structure. Numerical examples are used to demonstrate the rapid convergence of the method.

INTRODUCTION

Dynamic response analyses are important in the design of elastic structures subjected to time-dependent applied forces, because the loads associated with these responses can exceed the loads from all other sources. The dynamic response of a structure is determined by solving its equations of motion. The present practice in the aerospace industry is to simulate the actual structure with a discrete coordinate model to obtain the equations of motion. The discrete coordinate equations of motion that result are coupled second-order differential equations. Since coupled differential equations are rather difficult to solve, they are transformed to a set of uncoupled differential equations. The transformation is accomplished using the mode shapes from a modal analysis, and the uncoupled equations are called modal equations. Thus, the most important use of data from a modal analysis is to simplify the dynamic response analysis.

The modal analysis of a discrete coordinate model of a structure having n degrees of freedom requires the solution of an nth-order matrix eigenvalue problem. A digital computer is usually used to solve the eigenvalue problem. The three most important items to consider when choosing a method to solve the eigenvalue problem on a computer are accuracy of the solution, computation time required, and computer storage required. There are many algorithms available for direct solution of the matrix eigenvalue problem [1]; however, as the number of degrees of freedom becomes large, the computation time and computer storage becomes excessive for most of these methods.

The technique most often used in the aerospace industry to reduce the computation time and computer storage to acceptable levels for modal analysis is the Rayleigh-Ritz method [2]. This method permits an approximation of the first m modes of the structure to be obtained by solving an mth-order eigenvalue problem rather than the original nth-order eigenvalue problem. Thus, when the number of modes desired is significantly less than the number of degrees of freedom of the model, a great deal of computer time and computer storage can be saved by using this method. The reduction in the order of the eigenvalue problem is accomplished by approximating the displacements of the discrete coordinates by a linear sum of m Rayleigh-Ritz coordinate vectors.

The theory on which the Rayleigh-Ritz method is based places only minor restrictions on the coordinate vectors that can be used. Thus, an analyst has great freedom in the choice of the coordinate vectors. When the coordinate vectors are derived from the modal properties of the components of the structure, the technique is called "modal coupling" or "component mode synthesis". The most well known method based on component modal properties is due to Hurty [3].

The inherent problem in using the Rayleigh-Ritz method is the accuracy of the approximate solutions. That is, in general it is necessary to use more Rayleigh-Ritz coordinate vectors than the number of modes desired in order to obtain sufficient accuracy. This procedure tends to defeat the purpose of the Rayleigh-Ritz method since the order of the eigenvalue problem is equal to the number of coordinate vectors rather than the number of modes desired.

The accuracy problem associated with the Rayleigh-Ritz method can be eliminated by applying the method repeatedly and using improved coordinate vectors each time. This procedure will converge to the exact modes since each application of the Rayleigh-Ritz method will yield a closer approximation of the exact modes of the structure.

Bajan, Feng, and Jaszlics [4] have suggested an iterative method and applied it using coordinate vectors derived from component modal properties. The iterative method presented here appears to converge more rapidly than the method of Bajan, Feng, and Jaszlics, at least for the example in their paper.

ANALYTICAL DEVELOPMENT

Consider a discrete coordinate model of a constrained structure having n degrees of freedom. The free, undamped equations of motion of such a model expressed in matrix notation are:

$$[M]\{\ddot{X}\} + [K]\{X\} = \{0\} \quad , \tag{1}$$

where $\{X\}$ is a vector of discrete coordinate displacements, $[M]$ is the mass matrix, and $[K]$ is the stiffness matrix. The time dependence in equation (1) can be removed by the transformation $\{X\} = e^{i\omega t}\{X\}$, since its solutions are harmonic in time. Thus, equation (1) becomes

$$\Big([K] - \omega^2 [M]\Big)\{X\} = \{0\} \quad . \tag{2}$$

Equation (2) is recognized as a matrix eigenvalue problem of order n whose eigenvectors, $[Y]$, are the mode shapes and whose eigenvalues, $[\omega^2]$, are the squares of the frequencies.

The first step in using the Rayleigh-Ritz technique [2] to obtain approximate solutions of equation (2) consists of selecting a complete sequence of coordinate vectors

$$\{Z\}_1, \{Z\}_2, \ldots \{Z\}_m, \ldots \{Z\}_n \quad , \tag{3}$$

which are linearly independent and satisfy the kinematic boundary conditions. The displacement vector $\{X\}$ is then expressed as a linear sum of the first m coordinate vectors

$$\{X\} = \sum_{j=1}^{m} q_u \{Z\}_j \quad . \tag{4}$$

In matrix notation, equation (4) can be expressed as

$$\{X\} = [Z]\{q\} \quad . \tag{5}$$

It is known from energy considerations [2] that the function

$$\omega^2 = \{X\}^T [K]\{X\}/\{X\}^T [M]\{X\} \tag{6}$$

must be stationary. The equations that will yield ω^2 stationary are obtained by expressing equation (6) in terms of $\{q\}$, differentiating it with respect to the Rayleigh-Ritz coordinates, q_j, and setting the result equal to 0; i. e.,

$$\frac{\partial \omega^2}{\partial q_j} = 0 \quad , \quad j = 1 \text{ to } m. \tag{7}$$

Performing the operations indicated by equation (7) yields

$$\Big([Z]^T [K][Z] - \omega^2 [Z]^T [M][Z]\Big)\{q\} = \{0\} . \tag{8}$$

Equation (8) is a matrix eigenvalue problem of order m whose eigenvectors are $[y^*]$ and whose eigenvalues are $[\omega^2]$. The approximate mode shapes are obtained from the relationship

$$[Z] = [Z][Y^*] \quad . \tag{9}$$

The accuracy of the mode shapes and frequencies obtained from the Rayleigh-Ritz method depends entirely on the coordinate vectors, $[Z]$. In general, there are no procedures available for choosing a set of coordinate vectors that will yield exact solutions. However, it is possible to state characteristics such a set must possess. Since the mode shapes derived from the method are a linear combination of the coordinate vectors, it is clear that exact modes would result if each coordinate vector was a linear combination of the first m mode shapes of the structure. Another way of stating this is that exact results would be obtained for the first m modes of the structure if the coordinate vectors did not contain any contribution of modes higher than m.

Consider an iterative procedure to eliminate the accuracy problem associated with the Rayleigh-Ritz technique. If the Rayleigh-Ritz technique was repeatedly applied and if each time it was applied the coordinate vectors used would yield more accurate results than the previous set of vectors, it is clear that the iteration procedure would converge to the exact solution. The only problem with the iteration procedure is generating an improved set of coordinate vectors; that is, a set of vectors which will yield more accurate results than the previous set of coordinate vectors. Before an improved set of coordinate vectors can be generated, it is necessary to establish the characteristics such a set must possess.

From the discussion of the characteristics that a perfect set of coordinate vectors must possess, it is clear that an improved set of vectors is a set in which each vector contains less contribution from the higher modes than the corresponding vector in the previous set. Since the approximate mode shapes obtained by the Rayleigh-Ritz technique are a linear sum of the

coordinate vectors, it is possible to state the characteristics of an improved set of coordinate vectors in terms of a set of approximate mode shapes; that is, a set of coordinate vectors will be an improved set with respect to the previous set of vectors, if each vector contains less contribution from the higher modes than the corresponding mode shape derived from the previous set of coordinate vectors. Thus, an improved set of coordinate vectors can be generated by suppressing the contribution of the higher modes in the approximate mode shapes. The procedure for suppressing the contribution of higher modes is well known; in fact, it is the basis of the matrix iteration method [5] of modal analysis.

Consider generating a vector $\{f\}$ from a vector $\{s\}$ by the following equation:

$$\{f\} = [K]^{-1}[M]\{s\} \quad . \tag{10}$$

The vector $\{s\}$ can be expressed as a linear sum of the mode shapes of the structure

$$\{s\} = \sum_{j=1}^{n} \{y\}_j b_j \quad . \tag{11}$$

Combining equations (10) and (11) yields

$$\{f\} = \sum_{j=1}^{n} [K]^{-1}[M]\{y\}_j b_j \quad . \tag{12}$$

Since $\{y\}_j$ and ω_j must satisfy equation (2), the following relationship exists:

$$[K]^{-1}[M]\{y\}_j = \frac{1}{\omega_j^2}\{y\}_j \quad . \tag{13}$$

Therefore, equation (12) can be written as

$$\{f\} = \sum_{j=1}^{n} \frac{b_j}{\omega_j^2}\{y\}_j \quad . \tag{14}$$

Comparing equations (11) and (14), it is evident that the contribution of the higher modes in $\{f\}$ is less than in $\{s\}$, since the magnitude of ω_j^2 increases with j. Thus, an improved set of coordinate vectors can be obtained from the relationship

$$[Z] = [K]^{-1}[M][Y] \quad . \tag{15}$$

It is not advisable to use equation (15) directly to compute $[Z]$ since $[K]^{-1}$ is, in general, a full matrix and requires a great deal of storage. Thus, equation (15) is replaced by two equations and $[Z]$ is determined by solving a set of simultaneous equations; that is,

$$[F] = [M][Y] \tag{16}$$

and

$$[F] = [K][Z] \quad . \tag{17}$$

Equations (16) and (17) can be solved efficiently for structures whose mass and stiffness matrices are banded by using methods based on Cholesky's square-root technique [6].

The equations necessary in applying the iterative Rayleigh-Ritz technique have been developed. The essential steps of the technique can be summarized as follows:

1. Select a set of coordinate vectors $[Z]$.

2. Form the eigenvalue problem expressed by equation (8).

3. Solve equation (8) to yield the eigenvectors $[Y^*]$ and eigenvalues $[\omega^2]$.

4. Compute a set of approximate mode shapes by equation (9).

5. Compute an improved set of coordinate vectors $[Z]$ by solving equations (16) and (17).

6. Return to step 2.

The iteration is halted when the mode shapes and frequencies have attained the desired accuracy.

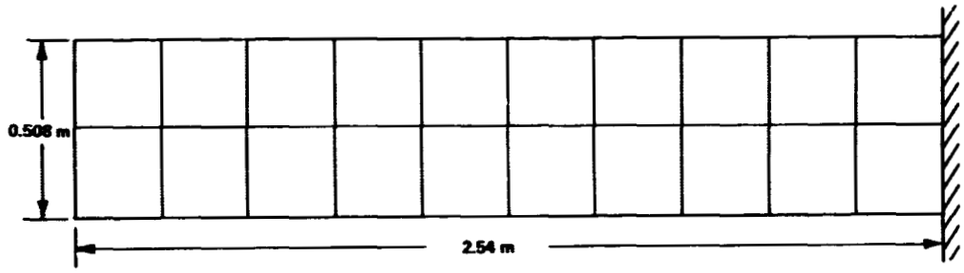
NUMERICAL RESULTS

The preceding analysis was programmed for an IBM 7094 digital computer. Included in the program was a provision for computing static deflections resulting from unit loads applied at selected discrete coordinates. These static deflections were used as the initial coordinate vectors for the analysis.

Modal data were generated for the cantilever frame structure shown in Figure 1. The structure is composed of 50 identical members, each of which can deform axially and in bending. The axial stiffness, AE, bending stiffness, EI, and the mass density, ρ, of the members are also shown in Figure 1. A discrete coordinate model of the structure was generated using 3 degrees of freedom at each junction of the members, which resulted in a total of 90 degrees of freedom. The degrees of freedom used at the junctions were a rotation, a horizontal displacement, and

a vertical displacement. Modal data for this model were computed using a standard eigenvalue routine to verify the results obtained from the iterative Rayleigh-Ritz technique. The first three modes of vibration of this structure are primarily bending modes, and the fourth mode is entirely an axial mode.

Figure 2 shows a plot of the square of the frequency versus the number of iterations. Also



THE DISCRETE COORDINATE MODEL OF THIS STRUCTURE HAS 3 DEGREES OF FREEDOM AT EACH JUNCTION FOR A TOTAL OF 90 DEGREES OF FREEDOM

BENDING STIFFNESS, $EI = 1.722 \times 10^2$ N/m^2

AXIAL STIFFNESS, $AE = 2.135 \times 10^5$ N

MATERIAL DENSITY, $\rho = 1.069 \times 10^4$ N-sec^2/m^4

Figure 1. Frame structure.

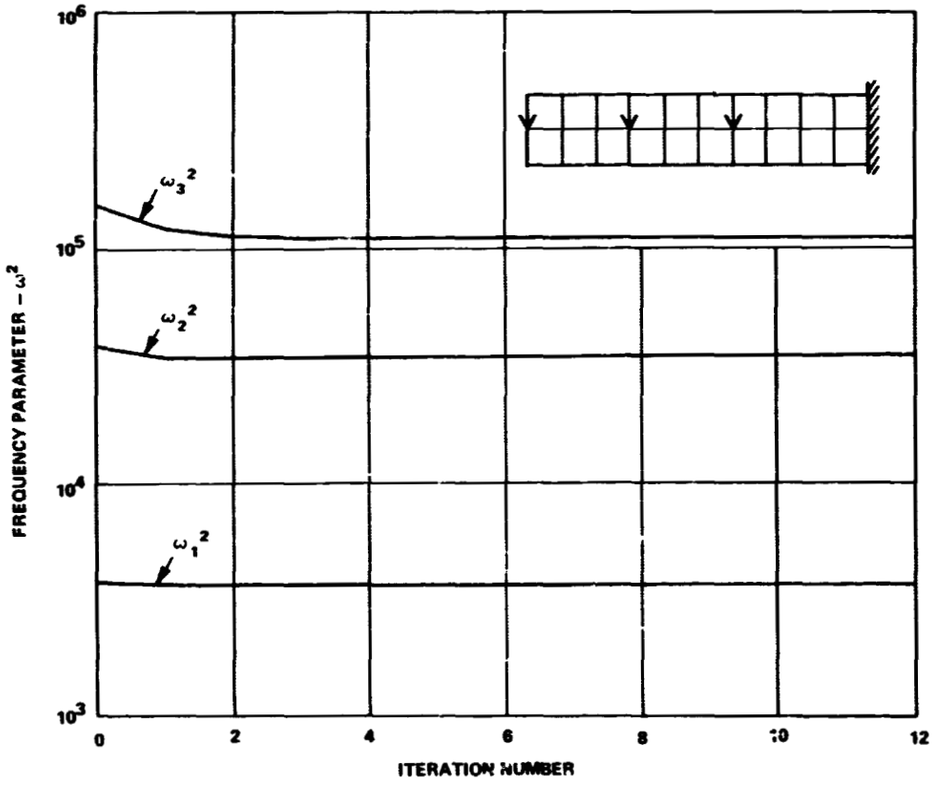


Figure 2. Plot of the square of the frequency versus the number of iterations.

shown in this figure is a sketch showing the structure with three arrows on it indicating the locations at which unit loads were applied to obtain the coordinate vectors. Since three coordinate vectors were used, three frequencies were obtained. They are shown plotted with a subscript indicating the mode number. The values of the frequencies shown for the 0th iteration are the frequencies computed from the initial set of coordinate vectors. All three frequencies converged to a constant value after three iterations with the third frequency requiring the most iterations since it was initially in error the most. From Figure 2 one could conclude that if the initial frequencies are not greatly in error, the technique produces rapid convergence. The next question that must be answered is: if the initial frequencies are greatly in error, will the technique still converge to the correct solution rapidly?

The purpose of Figure 3 is to show the rate of convergence of the frequencies for the case where the initial frequencies are greatly in error. This figure is similar to Figure 2 except that a different set of initial coordinate vectors was used. The locations of the unit loads used to generate the coordinate vectors are shown as arrows in the figure. These initial coordinate vectors yielded frequencies that were as much as two orders of magnitude in error. The technique still yielded accurate solutions after only four iterations. Thus, one could conclude from Figures 2 and 3 that the number of iterations required to obtain accurate solutions does not depend greatly on the initial set of coordinate vectors.

Thus far, only the convergence of the frequencies has been discussed. The rate of convergence of the mode shapes is also important. Figure 4 shows the rate of convergence of the mode shape corresponding to the third frequency shown in Figure 3. This mode was selected since its frequency converged the slowest of the three shown

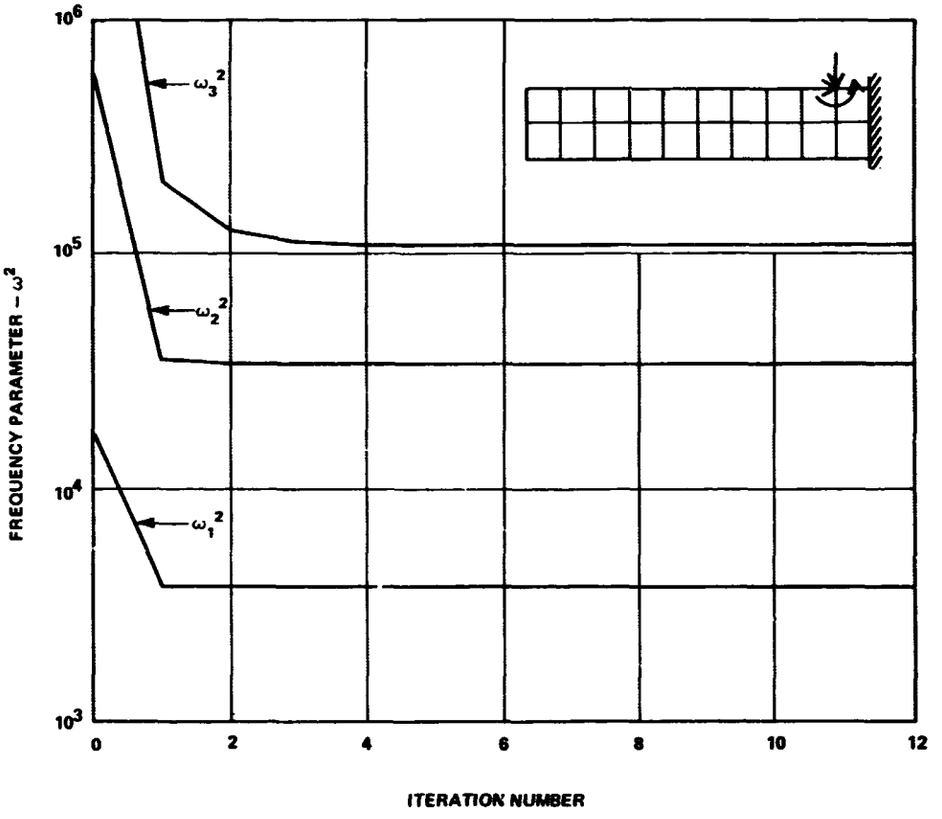


Figure 3. Rate of convergence of frequencies where the initial frequencies are greatly in error.

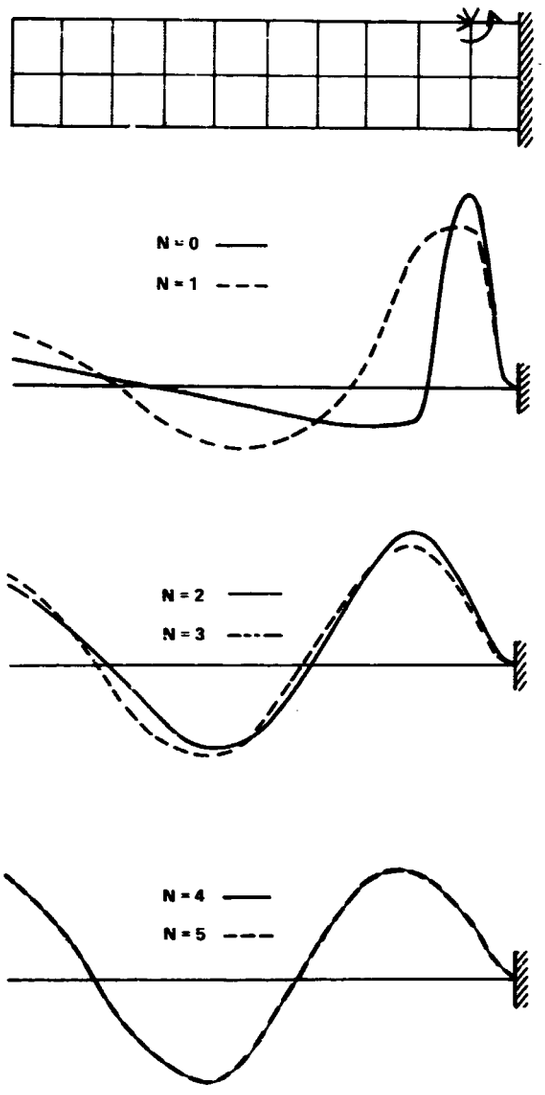


Figure 4. Mode shape of third mode.

in Figure 3. Figure 4 shows a plot of the centerline mode shape for iterations from zero to five. The mode shapes for the fourth and fifth iterations are almost identical. This indicates that the mode converged to a very accurate value in four iterations. Thus, from the numerical examples shown, it is evident that the iteration techniques will yield accurate mode shapes and frequencies after very few iterations.

Figure 5 shows what can happen if the coordinate vectors are generated arbitrarily. Again, the squares of the frequencies are plotted versus the iteration number, and the static loads used to generate the coordinate vectors are shown on the sketch. In the previous examples, all of the static loads used to generate the coordinate vectors caused the structure to deform at least to some extent in bending. That is not the case in this example. The horizontal force shown in Figure 5 causes no deformation normal to the centerline of the structure. Thus, the coordinate vector obtained from the horizontal force gives no approximation of the first three modes since they are bending modes. Figure 5 shows that the frequencies converged to a constant value rapidly, but the value to which the third frequency converged is not the true third frequency but is the fourth frequency of the structure. As mentioned previously, the fourth mode is an axial mode. Thus, the coordinate vectors chosen must approximate, at least to some extent, the type of modes that are being computed. After eight iterations, roundoff errors caused this mode to pick up a small amount of bending. When this happened, the iteration technique amplified this bending and suppressed the axial deformation, and it quickly converged to the true third frequency of the structure. Thus, some care must be taken in the coordinate vectors.

CONCLUSIONS

Based on the numerical examples presented, it appears the iterative Rayleigh-Ritz technique is an efficient means of computing modal data for structures having banded mass and stiffness matrices if reasonable care is used in selecting the coordinate vectors.

Initial coordinate vectors obtained from static loads were only used to demonstrate the iteration technique, and it should not be concluded that they are the best ones to be used. It is most likely that initial coordinate vectors derived from the modal properties of the components of the structure would be a better choice.

Even though the technique as presented applies only to constrained structures, it can be applied to unconstrained structures with only minor modifications.

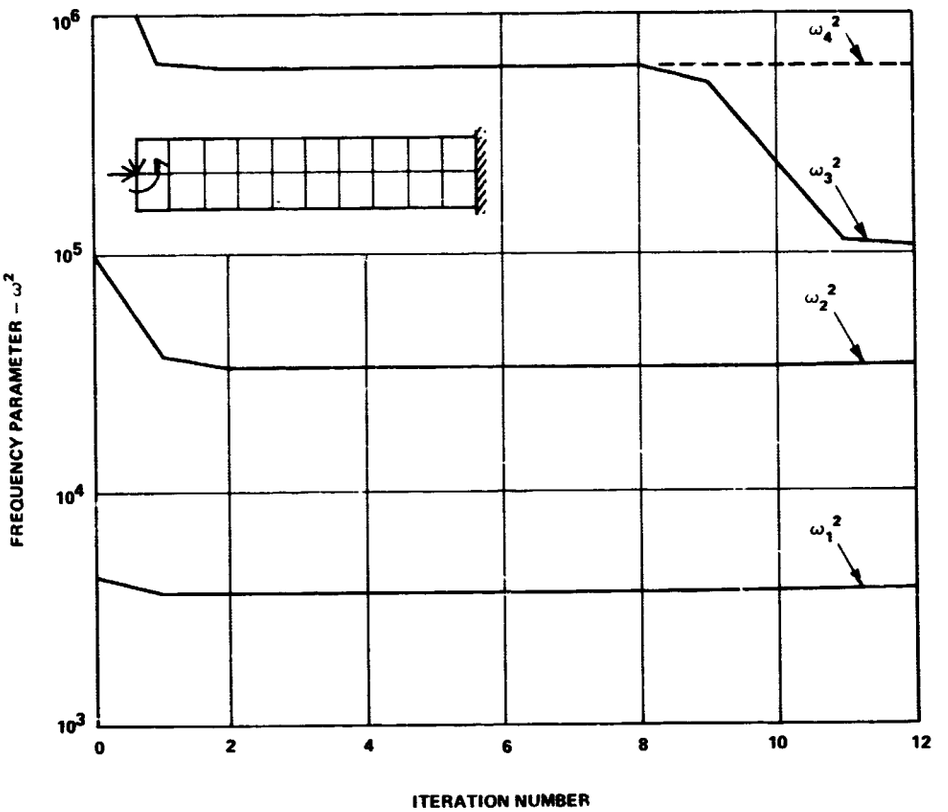


Figure 5. Plot of the square of the frequency versus the number of iterations with the coordinate vectors generated arbitrarily.

The most logical extension of this work would be to apply the iterative technique using initial coordinate vectors derived from the modal properties of the structure's components and generate the improved coordinate vectors using substructuring techniques.

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